Problem 1

a. (8 pts) Find a polar equation for the circle $(x-1)^2+y^2=1$.

$(x-1)^2+y^2=1$ $\qquad x = r\cos\theta$

$\qquad\qquad\qquad\qquad y = r\sin\theta$

$(r\cos\theta-1)^2+r^2\sin^2\theta=1$

$r^2\cos^2\theta+\cancel{1}-2r\cos\theta+r^2\sin\theta=\cancel{1}$

$r^2(\cos^2\theta+\sin^2\theta)=2r\cos\theta$

$r^2=2r\cos\theta$

$r(r-2\cos\theta)=0$

$r=0$ or $r-2\cos\theta=0$

$r=2\cos\theta$

$r=0 (=) 0(0,0)$ $\quad$ for $(r,\theta)=(0,\frac{\pi}{2}+k\pi;\ k\in\mathbb{Z})$ $\quad \theta=0$

So $\theta\in r=2\cos\theta$

So $r=2\cos\theta$ is a polar equation of the circle $(x-1)^2+y^2=1$

b. (8 pts) Find a Cartesian equation for the curve $r=\dfrac{1}{3\cos\theta-2\sin\theta}$.

Identify the curve.

$r=\dfrac{1}{3\cos\theta-2\sin\theta}$

$\Leftrightarrow 3r\cos\theta-2r\sin\theta=1$ $\quad$ with $3\cos\theta-2\sin\theta\neq0$

$3\cos\theta\neq2\sin\theta$

$\cancel{\tan\theta\neq\frac{3}{2}}$

$\cancel{\frac{y}{x}\neq\frac{3}{2}}$

$y\neq\frac{3}{2}x$

$3r\cos\theta\neq2r\sin\theta$

$3x\neq2y$

$3x-2y\neq0$

$3x-2y=1$

$3x-2y=1$ and $3x-2y=0$ are parallel so

they have no intersection points therefore

all points of $3x-2y=1$ are in the domain

of $r=\dfrac{1}{3\cos\theta-2\sin\theta}$ which is therefore a line

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Problem 2

Consider the function $f(x,y)=\sqrt{9-x^2-y^2}$

a. (6 pts) Find the domain and range of the function f.

$f(x,y)$ is defined for $9-x^2-y^2 \geq 0$

$x^2+y^2 \leq 9$

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So the domain is the inside of the circle (C) of center O(0,0) and radius $\sqrt{9}=3$ including the circle itself

b. (6 pts) Determine if the domain of f is an open region, a closed region, or neither and decide if the domain of f is bounded or unbounded.

The boundary points of the domain are the points of the circle which are in the domain so the domain is closed

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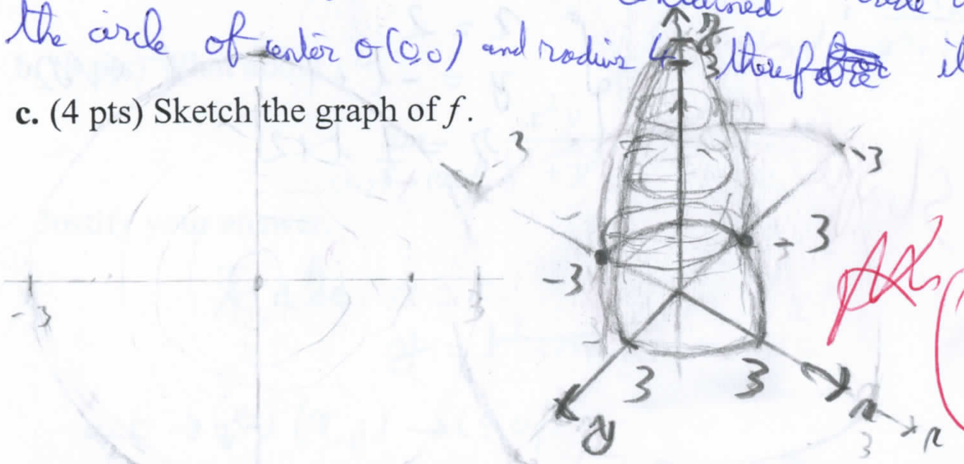
The domain can be contained inside a bigger figure like the circle of center O(0,0) and radius 4 therefore it is bounded

c. (4 pts) Sketch the graph of f.



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d. (6 pts) The plane $x=2$ intersects the graph of f at a curve C. Find the slope of the line L tangent to C in the plane $x=2$ at the point (2,1,2). Take $f(x,y,z)=\sqrt{9-x^2-y^2}-z$

~~The tangent plane at (2,1,2) on l then~~ $\vec{\nabla} f|_{(2,1,2)}$

The line L is tangent to C at (2,1,2) so $L \cdot \vec{\nabla} f|_{(2,1,2)}=0$

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$L \in$ plane $x=2$ of normal vector $\vec{i}(1,0,0)$ so $L \cdot \vec{i}=0$

So the directional vector $\vec{u}$ of L is $\vec{\nabla} f|_{(2,1,2)} \wedge \vec{i}$

$\frac{\partial f}{\partial x}\Big|_{(2,1,2)} = -\frac{2x}{2\sqrt{9-x^2-y^2}} = -\frac{2}{\sqrt{9-5}} = -1$

$\vec{\nabla} f|_{(2,1,2)}\left(-1,-\frac{1}{2},-1\right)$

$\frac{\partial f}{\partial y}\Big|_{(2,1,2)} = \frac{-2y}{2\sqrt{9-x^2-y^2}} = -\frac{1}{\sqrt{9-5}} = -\frac{1}{2}$

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$$\vec{n} = \nabla \vec{f}\Big|_{(2,1,2)} \cdot \vec{u} = \cancel{+2\vec{j} - \vec{k}}$$

$$\begin{vmatrix} \vec{i} & -2 & 1 \\ \vec{j} & -1 & 0 \\ \vec{k} & -1 & 0 \end{vmatrix}$$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \end{array}$$
$$\begin{vmatrix} -1 & -\frac{1}{2} & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -\vec{j} + \frac{1}{2}\vec{k}$$

So $\vec{u}\ (0, -1, \frac{1}{2})$ is a directional vector of L

$M(2,1,2) \in L$

So $N(x,y,z) \in L$ iff

$$\vec{MN} = t\,\vec{u} \quad t \in \mathbb{R} \iff \begin{cases} x-2 = 0 \\ y-1 = -t \\ z-2 = \frac{1}{2}t \end{cases} \quad t \in \mathbb{R}$$

$$\begin{cases} x = 2 \\ y = -t+1 \\ z = \frac{1}{2}t + 2 \end{cases} \quad t \in \mathbb{R}$$

slope?

Problem 3

a. (8 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+|y|}$$

exist? Justify your answer.

$$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{x^2 y}{(x^4+|y|)}\right| \le \lim_{(x,y)\to(0,0)} \frac{|x^2 y|}{|y|}$$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \le \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{|y|}$$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \le \lim_{(x,y)\to(0,0)} x^2$$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \le 0$$

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By sandwich theorem $\lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} = 0$

b. (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} \, ?$$

Justify your answer.

Take $x = t$

$y = \cancel{t^2+mt}\ mt^2$

as $t \to 0 \Leftrightarrow (x,y) \to (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = \lim_{t\to 0} \frac{t^2 \cdot mt^2}{t^4+m^2t^4} = \lim_{t\to 0} \frac{m\,t^4}{t^4(1+m^2)}$$

$$= \lim_{t\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

8

which depends on m

By the path test $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ doesn't exist no change

$$\vec{n} = \vec{\nabla f}\Big|_{(2,1,2)} \cdot \vec{u} = \cancel{+2\vec{j} - \vec{k}}$$

$$\begin{vmatrix} \vec{i} & -2 & 1 \\ \vec{j} & -1 & 0 \\ \vec{k} & -1 & 0 \end{vmatrix}$$

$$= -\vec{j} + \frac{1}{2}\vec{k}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -\frac{1}{2} & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

So $\vec{u}\,(0, -1, \frac{1}{2})$ is a directional vector of L

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slope?

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$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \leq \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{|y|}$$

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \leq \lim_{(x,y)\to(0,0)} x^2$$

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} \leq 0$$

By sandwich theorem $\lim_{(x,y)\to(0,0)} \frac{x^2|y|}{x^4+|y|} = 0$

8

b. (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} ?$$

Justify your answer.

Take $x=t$

$y = \cancel{t^2+mt}\ mt^2$

as $t\to 0 \Leftrightarrow (x,y)\to(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{t\to 0} \frac{t^2\cdot mt^2}{t^4+m^2t^4} = \lim_{t\to 0} \frac{m\,t^4}{t^4(1+m^2)}$$

$$= \lim_{t\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

which depends on m

8

By the path test $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ doesn't exist

no change

$proj_x\, v = \frac{x \cdot v}{v \cdot v} \cdot v$

Problem 4

(14 pts) Find the tangent plane and normal line of the surface $5x^2 + 4e^y = 12 - ln(z^3 x)$ at the point $(1, 0, e)$.

$P(1, 0, e)$

$5x^2 + 4e^y - 12 + \ln(z^3 x) = 0$

we Take $f(x,y,z) = 5x^2 + 4e^y - 12 + \ln(z^3 x) = 0$

we get $\vec{\nabla} f \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\frac{\partial f}{\partial x} = 10x + \frac{z^3}{z^3 x} = 10x + \frac{1}{x}$

$\frac{\partial f}{\partial y} = 4e^y$

$\frac{\partial f}{\partial z} = \frac{x \cdot 3z^2}{z^3 x} = \frac{3z^2}{z^3} = \frac{3}{z}$

~~At the~~

$\frac{\partial f}{\partial x}\Big|_{(1,0,e)} = 10 + \frac{1}{1} = 11$

$\frac{\partial f}{\partial x}\Big|_{(1,0,e)} = 4e^0 = 4$

$\frac{\partial f}{\partial x}\Big|_{(1,0,e)} = \frac{3}{e}$ $\quad \vec{\nabla} f\Big|_{(1,0,e)} \left(11, 4, \frac{3}{e}\right)$

$\vec{\nabla} f\Big|_{(1,0,e)}$ is the normal vector of the tangent plane (T) at $(1,0,e)$

on $f(x,y,z) = 0$

$Q(x,y,z) \in (T)$ iff

$\vec{PQ} \cdot \vec{\nabla} f\Big|_{(1,0,e)} = 0$

$\begin{pmatrix} x-1 \\ y \\ z-e \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ \frac{3}{e} \end{pmatrix}$

$11(x-1) + 4y + \frac{3}{e}(z-e) = 0$

$(T)\ 11x + 4y + \frac{3}{e} z - 14 = 0$

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The normal line (d) has $\vec{\nabla f}\big|_{(1,0,e)}$ as directional vector

So $N(x,y,z) \in (d)$ iff

$$\vec{PN} = t\,\vec{\nabla f}\big|_{(1,0,e)} \quad t \in \mathbb{R}$$

$$\begin{cases} x-1 = 11t \\ y = 4t \\ z-e = \frac{3}{e}t \end{cases} \quad t \in \mathbb{R} \iff (d): \begin{cases} x = 11t+1 \\ y = 4t \\ z = \frac{3}{e}t + e \end{cases} \quad t \in \mathbb{R}$$

Problem 5

(14 pts) Let $f(x, y, z)$ be a differentiable function of three variables. Suppose that $f(8,3,4) = 4$, $\nabla f(8,3,4) = 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$,

Let $\begin{cases} x = r^2 + s \\ y = \dfrac{s}{r} + 1 \\ z = 2r \\ \text{and} \quad w(r,s) = f(x, y, z) \end{cases}$

Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ at $(r,s) = (2,4)$.

$\vec{\nabla} f|_{(8,3,4)} = 4\vec{i} - 3\vec{j} - 7\vec{k}$

for $(r,s) = (2,4)$

$x = 2^2 + 4 = 8$

$y = \frac{4}{2} + 1 = 3$

$z = 2r = 4$

Then

$\frac{\partial f}{\partial x}\Big|_{(8,3,4)} = 4$

$\frac{\partial f}{\partial y}\Big|_{(8,3,4)} = -3$

$\frac{\partial f}{\partial z}\Big|_{(8,3,4)} = -7$

So by the chain rule

$$\frac{\partial w}{\partial r}\Big|_{(2,4)} = \frac{\partial f}{\partial x}\Big|_{(8,3,4)} \times \frac{\partial x}{\partial r}\Big|_{(2,4)} + \frac{\partial f}{\partial y}\Big|_{(8,3,4)} \frac{\partial y}{\partial r}\Big|_{(2,4)} + \frac{\partial f}{\partial z}\Big|_{(8,3,4)}$$

$$= 4 \times (2r + 2) + (-3)\left(-\frac{s}{r^2}\right) + (-7)(2)$$

$$= 4 \times 4 + (-3)(-1) + (-7)(2)$$

$$= 16 + 3 - 14 = 5$$

$$\frac{\partial w}{\partial s}\Big|_{(2,4)} = \frac{\partial f}{\partial x}\Big|_{(8,3,4)} \times \frac{\partial x}{\partial s}\Big|_{(2,4)} + \frac{\partial f}{\partial y}\Big|_{(8,3,4)} \times \frac{\partial y}{\partial s}\Big|_{(2,4)} + \frac{\partial f}{\partial z}\Big|_{(8,3,4)} \times \frac{\partial z}{\partial s}$$

$$= 4(1) + (-3)\left(\frac{1}{r}\right) + (-7)(0)$$

$$= 4 - \frac{3}{2} = \frac{5}{2}$$

Problem 6

(18 pts) Consider the function $f(x)=\left(1-\frac{x}{2}\right)^{-2}$ Find the Taylor polynomials $p_1(x)$ and $p_2(x)$ generated by f at the center $a=0$. Then use Taylor's theorem to estimate the error resulting from the approximation

$$f(x)\approx p_2(x) \quad \text{for} \quad 0\le x\le 0.1.$$

$(1-\frac{x}{2})^{-2}$ at the center $a=0$ $= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\cdot x^n$

$f(x)=(1-\frac{x}{2})^{-2}$

~~$f'(x)=-2\cdot -\frac{1}{2}(1-\frac{x}{2})^{-3}=-(1-\frac{x}{2})^{-3}$~~

~~$f''(x)=$~~

~~Let's take $(1-x)^{-2}$ at $a=0=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n$~~

$(1-x)$

$f^{(n)}$

$f(x)=(1-\frac{x}{2})^{-2}$

$f'(x)=-2\cdot -\frac{1}{2}\cdot(1-\frac{x}{2})^{-3}=(1-\frac{x}{2})^{-3}$

$f''(x)=-2\cdot(-\frac{1}{2})^2\cdot -3\,(1-\frac{x}{2})^{-4}=\frac{3}{2}(1-\frac{x}{2})^{-4}$

$f'''(x)=-4!\,(-\frac{1}{8})^3\,(1-\frac{x}{2})^{-5}=3(1-\frac{x}{2})^{-5}$

$f^{(n)}(x)=(-1)^n(n+1)!\cdot(-\frac{1}{2})^n\,(1-\frac{x}{2})^{-(n+2)}$

~~7~~ 4

$f^{(n)}(0)=(n+1)!\cdot(\frac{1}{2})^n\cdot 1^n=(n+1)!\,(\frac{1}{2})^n$

So $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n=\sum_{n=0}^{\infty}\frac{(n+1)!\,(\frac{1}{2})^n}{n!}x^n=\sum_{n=0}^{\infty}(n+1)\cdot(\frac{x}{2})^n$

$(1-\frac{x}{2})^{-2}=1+\frac{x}{2}+\frac{3}{4}x^2+\frac{1}{2}x^3+\ \text{———}$

(-2)

$=\frac{3}{3!}$

$p_1(x)=1+\frac{x}{2}$ -1

$p_2(x)=1+\frac{x}{2}+\frac{3}{4}x^2$

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By taylor's theorem using $p_2(x)$ as an estim of $f(x)$ gives an error of

$\frac{f'''(c)}{3!}\cdot x^3$ with $0\le c\le x\le 0.1$

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$0 \le c \le 0{,}1$

$-0{,}1 \le -\frac{c}{2} \le 0$

$(1-\frac{c}{2})^{-} \le 1-0$

$- \le c \le 0{,}1$

$1-\frac{c}{2} \ge 0{,}95$

$(1-\frac{c}{2})^{-5} \le 0{,}95^{-5}$

3

3

So $\frac{3(1-\frac{c}{2})^{-5}}{3!} \cdot x^3 \le \frac{3 \times 0{,}95^{-5}}{3!} \cdot x^3$

$\frac{3(1-\frac{c}{2})^{-5}}{3!}$ ~~error~~ $\le \frac{0{,}95^{-5}}{2} \cdot x^3$

$|R_2(x)| \le (0{,}1)^3$ -1

So by Taylor's theorem error is at most $= \frac{0{,}95^{-5}}{2} \cdot x^3$ for $0 \le x \le 0{,}1$

for the estimate $f(x) = P_2(x)$

$N(S) = 2$

$R(S) = 2$

$T: \mathbb{R}^3 \to \mathbb{R}^4$

$S: \mathbb{R}^4 \to \mathbb{R}^2$

$S \circ T: \mathbb{R}^3 \to \mathbb{R}^2$

$S \circ T = 0$

$N(S) = R(T)$